

Controllability and optimal control of the transport equation with a localized vector field*

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Abstract—We study controllability of a Partial Differential Equation of transport type, that arises in crowd models. We are interested in controlling such system with a control being a Lipschitz vector field on a fixed control set ω .

We prove that, for each initial and final configuration, one can steer one to another with such class of controls only if the uncontrolled dynamics allows to cross the control set ω .

We also prove a minimal time result for such systems. We show that the minimal time to steer one initial configuration to another is related to the condition of having enough mass in ω to feed the desired final configuration.

I. INTRODUCTION

In recent years, the study of systems describing a crowd of interacting autonomous agents has draw a great interest from the control community (see e.g. the Cucker-Smale model [4]). A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. Beside the description of interaction, it is now relevant to study problems of **control of crowds**, i.e. of controlling such systems by acting on few agents, or with a control localized in a small subset of the configuration space.

Two main classes are widely used to model crowds of interacting agents. In **microscopic models**, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. In **macroscopic models**, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. This is an example of a **distributed parameter system**. Some nonlocal terms can model the interactions between the agents. In this article, we focus on this second approach.

To our knowledge, there exist few studies of control of this kind of equations. In [7], the authors provide approximate alignment of a crowd described by the Cucker-Smale model [4]. The control is the acceleration, and it is localized in a control region ω which moves in time. In a similar

situation, a stabilization strategy has been established in [2], by generalizing the Jurdjevic-Quinn method to distributed parameter systems.

In this article, we study a partial differential equation of transport type, that is widely used for modeling of crowds. Let ω be a nonempty open connected subset of \mathbb{R}^d ($d \geq 1$), being the portion of the space on which the control is allowed to act. Let $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field assumed Lipschitz and uniformly bounded. Consider the following linear transport equation

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u) \mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \quad (1)$$

where $\mu(t)$ is the time-evolving measure representing the crowd density, and μ^0 is the initial data. The control function $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ satisfies $\text{supp}(u) \subset \bar{\omega}$. The function $v + \mathbb{1}_\omega u$ represents the velocity field acting on μ . System (1) is a first simple approximation for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand controllability properties for such simple equation. Indeed, the results contained in this article will be instrumental to a forthcoming paper, where we will study more complex crowd models, with a non-local term $v[\mu]$.

We now recall the precise notion of approximate controllability and exact controllability for System (1). We say that:

- System (1) is *approximately controllable* from μ^0 to μ^1 on the time interval $(0, T)$ if for each $\varepsilon > 0$ there exists u with $\text{supp}(u) \subset \bar{\omega}$ such that the corresponding solutions to System (1) satisfies $W_p(\mu(T), \mu^1) \leq \varepsilon$.
- System (1) is *exactly controllable* from μ^0 to μ^1 on the time interval $(0, T)$ if there exists a control u satisfying $\text{supp}(u) \subset \bar{\omega}$ such that the solution to System (1) is equal to μ^1 at time T .

The definition of the Wasserstein distance W_p is recalled in Section II.

To control System (1), from a geometrical point of view, the uncontrolled vector field v needs to send the support of μ^0 to ω forward in time and the support of μ^1 to ω backward in time. This idea is formulated in the following Condition:

Condition 1 (Geometrical condition) Let μ^0, μ^1 be two probability measures on \mathbb{R}^d satisfying:

- For all $x^0 \in \text{supp}(\mu^0)$, there exists $t_0 > 0$ such that $\Phi_{t_0}^v(x^0) \in \omega$, where Φ_t^v is the flow associated to v , i.e. the solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = v(x(t)) & \text{for a.e. } t > 0, \\ x(0) = x^0. \end{cases}$$

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- (ii) For all $x^1 \in \text{supp}(\mu^1)$, there exists $t_1 < 0$ such that $\Phi_{t_1}^v(x^1) \in \omega$.

Remark 1 Condition 1 is the minimal one that we can expect to steer any initial condition to any targets. Indeed, if the first item of Condition 1 is not satisfied, there exists a whole subpopulation of the measure μ_0 that never intersects the control region. Thus, we cannot act on it, and we cannot steer it to any desired target.

We denote by \mathcal{U} the set of admissible controls, that are functions $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ Lipschitz in space, measurable in time and uniformly bounded with $\text{supp}(u) \subset \bar{\omega}$. If we impose the classical Carathodory condition of u being in \mathcal{U} then the flow $\Phi_t^{v+1_\omega u}$ is an homeomorphism (see [1, Th. 2.1.1]). As a result, one cannot expect exact controllability, since for general measures there exists no homeomorphism sending one to another. We then have the following result of approximate controllability.

Theorem 1 *Let μ^0, μ^1 two probability measures on \mathbb{R}^d with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1. Then there exists $T > 0$ such that System (1) is approximately controllable at time T with a control u in \mathcal{U} .*

The proof of this result will be given in Section III. After having proved approximate controllability for (1), we aim to study the minimal time problem, i.e. the minimal time to send μ_0 to μ_1 . We have the following result.

Theorem 2 *Let μ^0, μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.*

Define T^ an admissible time if it satisfies*

- (a) *For each $x^0 \in \text{supp}(\mu^0)$*

$$T^* \geq \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}.$$
- (b) *For each $x^1 \in \text{supp}(\mu^1)$*

$$T^* \geq \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}.$$
- (c) *There exists a sequence $(\theta_k)_k$ of $C^\infty(\Omega)$ -functions equal to 1 in ω^c such that*

$$\lim_{k \rightarrow \infty} [\Phi_t^{\theta_k v} \# \mu^0](\omega) \geq 1 - \lim_{k \rightarrow \infty} [\Phi_{t-T^*}^{\theta_k v} \# \mu^1](\omega), \quad (2)$$

Let T_0 be the infimum of such T^ . Then, for all $T > T_0$, System (1) is approximately controllable at time T .*

The proof of this Theorem is given in Section IV.

Remark 2 The meaning of condition (2) is the following: functions θ_k are used to store the mass in ω . Thus, condition (2) means that at each time t there is more mass that has entered ω than mass that has exited. This is the minimal condition that we can expect in this setting, since control can only move masses, without creating them.

This paper is organized as follows. In Section II, we recall some properties of the continuity equation and the Wasserstein distance. Sections III and IV are devoted to prove Theorems 1 and 2, respectively. We conclude with some numerical examples in Section V.

II. THE CONTINUITY EQUATION AND THE WASSERSTEIN DISTANCE

In this section, we recall some properties of the continuity equation (1) and of the Wasserstein distance, which will be used all along this paper.

We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in \mathbb{R}^d with compact support, and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of measures which are absolutely continuous with respect to the Lebesgue measure. First of all, we give the definition of the push-forward of a measure and of the Wasserstein distance.

Definition 1 Denote by Γ the set of the measurable maps $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$. For a $\gamma \in \Gamma$, we define the push-forward $\gamma \# \mu$ of a measure μ of \mathbb{R}^d as follows:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for all Borel sets E .

Definition 2 Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Define

$$W_p(\mu, \nu) = \inf_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}. \quad (3)$$

Then, W_p is a distance on $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, called the **Wasserstein distance**.

Moreover, the Wasserstein distance can be extended to all pairs of measures μ, ν with the same mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p \left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|} \right).$$

For more details about the Wasserstein distance, in particular for its definition on the whole space of measures $\mathcal{P}_c(\mathbb{R}^d)$, we refer to [8, Chap. 7].

We now recall a standard result for the continuity equation:

Theorem 3 ([8]) *Let $T \in \mathbb{R}$, $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and w be a vector field uniformly bounded, Lipschitz in space and measurable in time. Then the system*

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases} \quad (4)$$

admits a unique solution¹ μ in $C^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$. Moreover, it holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$ for all $t \in \mathbb{R}$, where the flow $\Phi_t^w(x^0)$ is the unique solution at time t to

$$\begin{cases} \dot{x}(t) = w(t, x(t)) \text{ for a.e. } t \geq 0, \\ x(0) = x^0. \end{cases} \quad (5)$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

Property 1 ([6]) *Let $w : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time. For each $t \in \mathbb{R}$, it holds*

$$W_p^p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leq e^{(p+1)L|t|} W_p^p(\mu, \nu), \quad (6)$$

where L is the Lipschitz constant of w .

¹Here, $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ is equipped with the weak topology, that coincides with the topology induced by the Wasserstein distance W_p , see [8, Thm 7.12].

Property 2 Let μ, ν, ρ, η some positive measures satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$. It then holds

$$W_p^p(\mu + \rho, \nu + \eta) \leq W_p^p(\mu, \nu) + W_p^p(\rho, \eta) \quad (7)$$

Using the properties of Wasserstein distance given in Section 1 of [6], we can replace W_p by W_1 in the definition of the approximate controllability.

III. PROOF OF THEOREM 1

In this section, we prove approximate controllability of System (1). The proof is based on three approximation steps, corresponding to Proposition 1, 2, and 3. The proof is then given at the end of the section.

In a first step, we suppose that ω contains the support of both μ^0, μ^1 .

Proposition 1 Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ be such that $\text{supp}(\mu^0, \mu^1) \subset\subset \omega$. Then, for all $T > 0$, System (1) is approximately controllable at time T with a control u in \mathcal{U} .

Proof: We assume that $d := 2, T := 1$ and $\omega = (0, 1)^2$, but the reader will see that the proof can be clearly adapted to any space dimension. Fix $n \in \mathbb{N}^*$. Define $a_0 := 0, b_0 := 0$ and the points a_i, b_i for all $i \in \{1, \dots, n\}$ by induction as follows: suppose that for $i \in \{0, \dots, n-1\}$ the points a_i and b_i are given, then a_{i+1} and b_{i+1} are the smallest values satisfying

$$\int_{(a_i, a_{i+1}) \times \mathbb{R}} d\mu^0 = \frac{1}{n} \quad \text{and} \quad \int_{(b_i, b_{i+1}) \times \mathbb{R}} d\mu^1 = \frac{1}{n}.$$

Again, for all $i \in \{0, \dots, n-1\}$, we define $a_{i,0} := 0, b_{i,0} := 0$ and supposing that for a $j \in \{0, \dots, n-1\}$ the points $a_{i,j}$ and $b_{i,j}$ are already defined, $a_{i,j+1}$ and $b_{i,j+1}$ is the smallest values such that

$$\int_{A_{ij}} d\mu^0 = \frac{1}{n^2} \quad \text{and} \quad \int_{B_{ij}} d\mu^1 = \frac{1}{n^2},$$

where $A_{ij} := (a_i, a_{i+1}) \times (a_{ij}, a_{i(j+1)})$ and $B_{ij} := (b_i, b_{i+1}) \times (b_{ij}, b_{i(j+1)})$. Since μ^0 and μ^1 have a mass equal to 1 and have support of $(0, 1)$, then $a_n, b_n \leq 1$ and $a_{i,n}, b_{i,n} \leq 1$ for all $i \in \{0, \dots, n-1\}$. We give in Figure 1 an example of such decomposition.

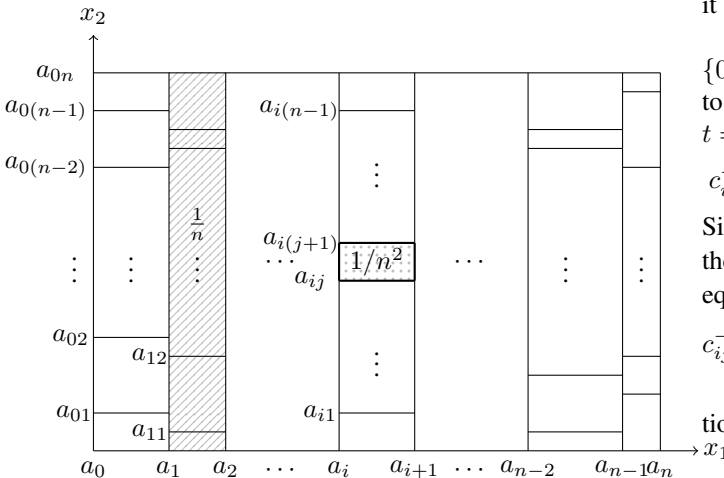


Fig. 1. Example of a decomposition of μ^0 .

If one aims to define a vector field sending each A_{ij} to B_{ij} , then some shear stress is naturally introduced. To overcome this problem, we first define sets $\tilde{A}_{ij} \subset A_{ij}$ and $\tilde{B}_{ij} \subset B_{ij}$ for all $i, j \in \{0, \dots, n-1\}$. We then send the mass of μ^0 from each \tilde{A}_{ij} to each \tilde{B}_{ij} , while we do not control the mass in $A_{ij} \setminus \tilde{A}_{ij}$. More precisely, for all $i, j \in \{0, \dots, n-1\}$, we define, as in Figure 2, $a_i^-, a_i^+, a_{ij}^-, a_{ij}^+$ the smallest values such that

$$\int_{(a_i, a_i^-) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \int_{(a_i^+, a_{i+1}) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \frac{1}{n^3},$$

and

$$\begin{aligned} \int_{(a_i^-, a_i^+) \times (a_{ij}, a_{ij}^-)} d\mu^0 &= \int_{(a_i^-, a_i^+) \times (a_{ij}^+, a_{i(j+1)})} d\mu^0 \\ &= \frac{1}{n} \times \left(\frac{1}{n^2} - \frac{2}{n^3} \right). \end{aligned}$$

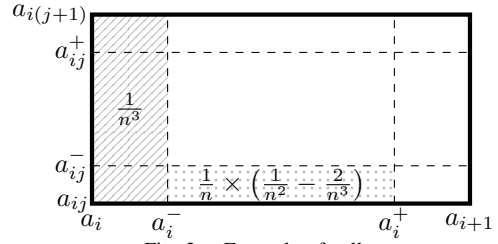


Fig. 2. Example of cell

We define similarly $b_i^+, b_i^-, b_{ij}^+, b_{ij}^-$. We finally define

$$\tilde{A}_{ij} := [a_i^-, a_i^+] \times [a_{ij}^-, a_{ij}^+] \quad \text{and} \quad \tilde{B}_{ij} := [b_i^-, b_i^+] \times [b_{ij}^-, b_{ij}^+].$$

The goal is to build a solution to System (1) such that the corresponding flow Φ_t^u satisfies

$$\Phi_T^u(\tilde{A}_{ij}) = \tilde{B}_{ij}, \quad (8)$$

for all $i, j \in \{0, \dots, n-1\}$. We observe that we do not take into account the displacement of the mass contained in $A_{ij} \setminus \tilde{A}_{ij}$. We will show that the corresponding term $W_1(\sum_{ij} \mu_{|A_{ij} \setminus \tilde{A}_{ij}}^0, \sum_{ij} \mu_{|B_{ij} \setminus \tilde{B}_{ij}}^1)$ tends to zero when n goes to the infinity. The rest of the proof is divided into two steps. In a first step, we build a flow and a velocity field such that its flow satisfies (8). In a second step, we compute the Wasserstein distance between μ^1 and $\mu(T)$ showing that it converges to zero when n goes to infinity.

Step 1: We first build a flow satisfying (8). For all $i \in \{0, \dots, n-1\}$, denote by c_i^- and c_i^+ the linear functions equal to a_i^- and a_i^+ at time $t = 0$ and equal to b_i^- and b_i^+ at time $t = 1$, respectively i.e.

$$c_i^-(t) = (b_i^- - a_i^-)t + a_i^- \quad \text{and} \quad c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+.$$

Similarly, for all $i, j \in \{0, \dots, n-1\}$, denote by c_{ij}^- and c_{ij}^+ the linear functions equal to a_{ij}^- and a_{ij}^+ at time $t = 0$ and equal to b_{ij}^- and b_{ij}^+ at time $t = 1$, respectively, i.e.

$$c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^- \quad \text{and} \quad c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+.$$

Consider the flow being the following the linear combination of c_i^-, c_i^+ and c_{ij}^-, c_{ij}^+ , i.e.

$$\begin{cases} \Phi_t^u(x^0)_1 = \frac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \frac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t), \\ \Phi_t^u(x^0)_2 = \frac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \frac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t), \end{cases} \quad (9)$$

when $x^0 \in A_{ij}$. We remark that Φ_t^u is piecewise- \mathcal{C}^1 and it solution to

$$\begin{cases} \frac{dx_{1t}}{dt} = \alpha_i(t)x_{1t} + \beta_i(t) & \forall t \in [0, T], \\ \frac{dx_{2t}}{dt} = \alpha_{ij}(t)x_{2t} + \beta_{ij}(t) & \forall t \in [0, T], \end{cases}$$

where for all $t \in [0, 1]$

$$\begin{cases} \alpha_i(t) = \frac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, \quad \beta_i(t) = \frac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\ \alpha_{ij}(t) = \frac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, \quad \beta_{ij}(t) = \frac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}. \end{cases}$$

For all $t \in [0, 1]$, consider the set $C_{ij}(t) := [c_i^-(t), c_i^+(t)] \times [c_{ij}^-(t), c_{ij}^+(t)]$. We remark that $C_{ij}(0) = \tilde{A}_{ij}$ and $C_{ij}(1) = \tilde{B}_{ij}$. On $C_{ij} := \{(x, t) : t \in [0, T], x \in C_{ij}(t)\}$, we then define the velocity field u by

$u_1(x, t) = \alpha_i(t)x_1 + \beta_i(t)$ and $u_2(x, t) = \alpha_{ij}(t)x_2 + \beta_{ij}(t)$, for all $(x, t) \in C_{ij}$. If we extend u by a \mathcal{C}^∞ and uniformly bounded function outside of $\cup_{ij} C_{ij}$, then $u \in \mathcal{U}$. Then, System (1) admits a unique solution and the flow on C_{ij} is given by the expression (9).

Step 2: We now prove that the refinement of the grid provides convergence to the target μ_1 , i.e.

$$W_1(\mu^1, \mu(1)) \xrightarrow{n \rightarrow \infty} 0. \quad (10)$$

We remark that

$$\int_{\tilde{B}_{ij}} d\mu(1) = \int_{\tilde{B}_{ij}} d\mu^1 = \frac{(n-2)^2}{n^4}.$$

Hence, by defining $R := (0, 1)^2 \setminus \bigcup_{ij} \tilde{B}_{ij}$, we also have

$$\int_R d\mu(1) = \int_R d\mu^1 = 1 - \frac{(n-2)^2}{n^2}.$$

We deduce that

$$\begin{aligned} W_1(\mu^1, \mu(T)) &\leq W_1(\mu^1 \times \mathbb{1}_R, \mu(1) \times \mathbb{1}_R) \\ &+ \sum_{i,j=1}^n W_1(\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}, \mu(1) \times \mathbb{1}_{B_{ij}}). \end{aligned} \quad (11)$$

We estimate each term in the right-hand side. Since we deal with absolutely continuous measures, the infimum in the definition of the Wasserstein distance is achieved, hence there exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, for all $i, j \in \{0, \dots, n-1\}$, and $\bar{\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\gamma_{ij} \# (\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}) = \mu(1) \times \mathbb{1}_{\tilde{B}_{ij}}$$

and

$$\bar{\gamma} \# (\mu^1 \times \mathbb{1}_R) = \mu(1) \times \mathbb{1}_R.$$

In the first term, for each $i, j \in \{0, \dots, n-1\}$, observe that γ_{ij} moves masses inside B_{ij} only. Thus

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}, \mu(T) \times \mathbb{1}_{\tilde{B}_{ij}}) &= \int_{\tilde{B}_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ &\leq (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4}. \end{aligned} \quad (12)$$

For the other term, observe that $\bar{\gamma}$ move a small mass in the bounded set ω . Thus it holds

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R) &\leq \int_R |x - \bar{\gamma}(x)| d\mu^1(x) \\ &\leq \sqrt{2} \left(1 - \frac{(n-2)^2}{n^2}\right) = 4\sqrt{2} \frac{n-1}{n^2}. \end{aligned} \quad (13)$$

We thus have (10) by combining (11), (12) and (13). \blacksquare

In the rest of the section, we remove the constraint $\text{supp}(\mu^0), \text{supp}(\mu^1) \subset \subset \omega$, now imposing Condition 1. First of all, we give a consequences of Condition 1.

Lemma 1 *If Condition 1 is satisfied, then the following Condition 2 is satisfied too:*

Condition 2 Let μ^0, μ^1 be two probability measures on \mathbb{R}^d . There exist two real numbers $T_0^*, T_1^* > 0$ such that

- (i) For all $x^0 \in \text{supp}(\mu^0)$, there exists $t_0 \leq T_0^*$ such that $\Phi_{t_0}^v(x^0) \in \omega$, where Φ_t^v is the flow associated to v .
- (ii) For all $x^1 \in \text{supp}(\mu^1)$, there exists $t_1 \in [-T_1^*, 0]$ such that $\Phi_{t_1}^v(x^1) \in \omega$.

Proof: Let us use an compactness argument. Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Assume that Condition 1 holds. Let $x^0 \in \text{supp}(\mu^0)$. Using Condition 1, there exists $t_0(x^0)$ such that $\Phi_{t_0(x^0)}^v(x^0) \in \omega$. Choose $r(x^0) > 0$ such that $B_{r(x^0)}(\Phi_{t_0(x^0)}^v(x^0)) \in \omega$, that exists since ω is open. By continuity of the application $x^1 \mapsto \Phi_{t_0(x^0)}^v(x^1)$ (see [1, Th. 2.1.1]), there exists $\hat{r}(x^0)$ such that

$$x^1 \in B_{\hat{r}(x^0)}(x^0) \Rightarrow \Phi_{t_0(x^0)}^v(x^1) \in B_{r(x^0)}(\Phi_{t_0(x^0)}^v(x^0)).$$

Since μ^0 is compactly supported, we can find a set $\{x_1^0, \dots, x_N^0\} \subset \text{supp}(\mu^0)$ such that

$$\text{supp}(\mu^0) \subset \bigcup_{i=1}^N B_{\hat{r}(x_i^0)}(x_i^0).$$

Thus the first item of Lemma 1 is satisfied for

$$T_0^* := \max\{t_0(x_i^0) : i \in \{1, \dots, N\}\}.$$

The proof of the existence of T_1^* is similar. \blacksquare

We now prove that we can store nearly the whole mass of μ_0 in ω , under Condition 2.

Proposition 2 *Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the first item of Condition 2. Then, for all $\delta, \varepsilon > 0$, there exists a space-dependent velocity field u Lipschitz and uniformly bounded such that the corresponding solution to System (1) satisfies*

$$\mu(T_0^* + \delta)(\omega) \geq 1 - \varepsilon. \quad (14)$$

Proof: Let $k \in \mathbb{N}^*$ and define $\theta_k \in \mathcal{C}^\infty(\Omega)$ a cutoff function on ω and the sets $\omega_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega^c) \geq 1/k\}$ satisfying

$$\begin{cases} 0 \leq \theta_k \leq 1, \\ \theta_k = 1 \text{ in } \omega^c, \\ \theta_k = 0 \text{ in } \omega_k. \end{cases} \quad (15)$$

Define

$$u_k := (\theta_k - 1)v. \quad (16)$$

We remark that the support of u is included in ω . Let $x^0 \in \text{supp}(\mu^0)$. Define

$$t_k(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega_k\}.$$

Consider the flow $y := \Phi_t^v(x^0)$ associated to x^0 without control, i.e. the solution to

$$\begin{cases} \dot{y}(t) = v(y(t)), \\ y(0) = x^0 \end{cases}$$

and the flow $z_k := \Phi_t^{u_k+v}(x^0)$ associated to x^0 with the control u_k given in (16), i.e. the solution to

$$\begin{cases} \dot{z}_k(t) = (v + u_k)(z_k(t)) = \theta(z_k(t)) \times v(z_k(t)), \\ z_k(0) = x^0. \end{cases} \quad (17)$$

Let us prove that the range of z_k for $t \geq 0$ is included in the range of y for $t \geq 0$. Consider the solution γ_k to the following system

$$\begin{cases} \dot{\gamma}_k(t) = \theta_k(y(\gamma_k(t))), \quad t \geq 0, \\ \gamma_k(0) = 0. \end{cases} \quad (18)$$

Since θ_k and y are Lipschitz, then System (18) admits a solution defined for all times. We remark that $\xi_k := y \circ \gamma_k$ is solution to System (17). Indeed for all $t \geq 0$

$$\begin{cases} \dot{\xi}_k(t) = \dot{\gamma}_k(t) \times \dot{y}(\gamma_k(t)) = \theta_k(\xi_k(t)) \times v(\xi_k(t)), \\ \xi_k(0) = y(\gamma_k(0)) = y(0). \end{cases}$$

By uniqueness of the solution to System (17), we obtain

$$y(\gamma_k(t)) = z_k(t) \text{ for all } t \geq 0.$$

Using the fact that $0 \leq \theta \leq 1$ and the definition of γ_k , we have

$$\begin{cases} \gamma_k \text{ increasing,} \\ \gamma_k(t) \leq t \quad \forall t \in [0, t_k(x^0)], \\ \gamma_k(t) \leq t_k(x^0) \quad \forall t \geq t_k(x^0). \end{cases}$$

We deduce that, for all $x^0 \in \text{supp}(\mu^0)$,

$$\{z_k(t) : t \geq 0\} \subset \{y(s) : s \in [0, t_k(x^0)]\}.$$

Thus, for K large enough it holds

$$\mu^0(\omega \setminus \omega_K) + \mu^0(S_K) \leq \varepsilon,$$

where

$$S_K := \{x^0 \in \text{supp}(\mu^0) \setminus \omega : \exists s \in (t_0, t_k), \Phi_s^v(x^0) \notin \bar{\omega}\}$$

and $t_0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}$. This implies that for $u := u_K$ the solution to System (1) satisfies (14). ■

The third step of the proof is to restrict a measure contained in ω to a measure contained in a square $S \subset \omega$.

Proposition 3 *Let $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying $\text{supp}(\mu^0) \subset \subset \omega$. Define S a square strictly included in ω and choose $\delta > 0$. Then there exists a space-dependent velocity field u , Lipschitz and uniformly bounded with $\text{supp}(u) \subset \bar{\omega}$, such that the corresponding solution to System (1) satisfies*

$$\text{supp}(\mu(\delta)) \subset \subset S.$$

Proof: From [5, Lemma 1.1, Chap. 1] and [3, Lemma 2.68, Chap. 2], there exists a function $\eta \in C^2(\bar{\omega})$ satisfying

$$\kappa_0 \leq |\nabla \eta| \leq \kappa_1 \text{ in } \omega \setminus S, \quad \eta > 0 \text{ in } \omega \quad \text{and} \quad \eta = 0 \text{ on } \partial\omega, \quad (19)$$

with $\kappa_0, \kappa_1 > 0$. We extend η by zero outside of ω . Let $k \in \mathbb{N}^*$ and the set ω_k defined in (15) such that $S \subset \subset \omega_k$. We denote by

$$u_k := k \nabla \eta - v.$$

Let $x^0 \in \text{supp}(\mu^0)$. Consider the flow $z_k(t) = \Phi_t^{v+u_k}(x^0)$ associated to x^0 , i.e. the solution to system

$$\begin{cases} \dot{z}_k(t) = v(z_k(t)) + u_k(z_k(t)) = k \nabla \eta(z_k(t)), \quad t \geq 0, \\ z_k(0) = x^0. \end{cases}$$

The different conditions in (19) imply that $n \cdot \nabla \eta < 0$ on $\partial\omega_0$, where n represents that exterior normal vector to $\partial\omega_0$. Thus $z_k(t) \in \omega$ for all $t \geq 0$.

We now prove that there exists K and $t \in [0, \delta]$ such that $z_K(t)$ belongs to the square S for all $x^0 \in \text{supp}(\mu^0)$. By contradiction, assume that for all $k \in \mathbb{N}^*$ and $t \in [0, \delta]$ there exists $x^0 \in \text{supp}(\mu^0)$ such that

$$z_k(t) \in S^c. \quad (20)$$

Consider the function f_k defined for all $t \in [0, \delta]$ by

$$f_k(t) := \eta(z_k(t)).$$

Its time derivative is given by

$$\dot{f}_k(t) = \dot{z}_k(t) \times \nabla \eta(z_k(t)) = k |\nabla \eta(z_k(t))|^2.$$

Then, using (20) and the property (19) of η , it holds

$$f_k(\delta) > k \kappa^2 \delta,$$

which is in contradiction with the fact that

$$f_k(\delta) \leq \|\eta\|_\infty.$$

Thus we deduce that, for a $K \in \mathbb{N}^*$ and a $t \in [0, \delta]$, $\Phi_t^{v+u_K}(x^0)$ belongs to S for all $x^0 \in \text{supp}(\mu^0)$. We conclude applying Proposition 2, replacing ω by S and v by $v + u_K$. ■

We now have all the tools to prove Theorem 1. The idea is the following: we first send μ_0 inside ω with a control u_1 , then from ω to a square S with a control u_2 . On the other side, we send μ_1 inside ω backward in time with a control u_5 , then from ω to S with a control u_4 . When both the source and the target are in S , we send one to the other with a control u_3 .

Proof of Theorem 1: Consider μ^0, μ^1 satisfying Condition 1. Then, by Lemma 1, there exist T_0^*, T_1^* for which μ^0, μ^1 satisfy Condition 2. Define $T := T_0^* + T_1^*$.

Choose $\delta, \varepsilon > 0$ and denote by $T_1 := T_0^* + \delta/5$, $T_2 := T_0^* + 2\delta/5$, $T_5 := T_1^* + \delta/5$ and $T_4 := T_1^* + 2\delta/5$. Using Propositions 2 and 3, there exists some controls u^1, u^2, u^4, u^5 Lipschitz and uniformly bounded and a square $S \subset \omega$ such that the solutions to

$$\begin{cases} \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^1) \rho_0) = 0 & \text{in } \mathbb{R}^d \times [0, T_1], \\ \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^2) \rho_0) = 0 & \text{in } \mathbb{R}^d \times [T_1, T_2], \\ \rho_0(0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases} \quad (21)$$

and

$$\begin{cases} \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^5) \rho_1) = 0 & \text{in } \mathbb{R}^d \times [-T_5, 0], \\ \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^4) \rho_1) = 0 & \text{in } \mathbb{R}^d \times [-T_4, -T_5], \\ \rho_1(0) = \mu^1 & \text{in } \mathbb{R}^d, \end{cases} \quad (22)$$

satisfy $\text{supp}(u_i) \subset \bar{\omega}$, $\rho_0(T_0^* + 2\delta/5)(S) > 1 - \varepsilon$ and $\rho_1(-T_1^* - 2\delta/5)(S) > 1 - \varepsilon$.

We now apply Proposition 1 to steer $\rho_0(T_0^* + 2\delta/5)$ to $\rho_1(-T_1^* - 2\delta/5)$ inside S : this gives a control u_3 on the time interval $[0, \frac{\delta}{5}]$. Thus, concatenating u_1, u_2, u_3, u_4, u_5 on the time interval $[0, T + \delta]$, we approximately steer μ_0 to μ_1 . ■

IV. PROOF OF THEOREM 2

In this section, we prove Theorem 2 about minimal time. To achieve controllability in this setting, one needs to store the mass coming from μ^0 in ω and to send it out with a rate adapted to approximate μ^1 .

Let T^* be the infimum satisfying Condition (2), and fix $s > 0$. We now prove that System (1) is approximately controllable at time $T := T^* + s$. Consider $N \in \mathbb{N}^*$, $\tau := T^*/N$, $\delta < \tau$, $\xi := \tau - \delta$ and $\tau_i := i \times \tau$. Define

$$\begin{cases} A_i := \{x^0 \in \text{supp}(\mu^0) : t_0(x^0) \in [0, \tau_i]\}, \\ B_i := \{x^1 \in \text{supp}(\mu^1) : T - t_1(x^1) \in [\tau_i, \tau_{i+1}]\}, \end{cases}$$

where

$$\begin{cases} t_0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}, \\ t_1(x^1) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}. \end{cases}$$

We remark that $\mu^0 \times \mathbb{1}_{A_i}$ represents the mass of μ^0 which has entered ω at time τ_i and $\mu^1 \times \mathbb{1}_{B_i}$ the mass of μ^1 which need to exit ω in the time interval (τ_i, τ_{i+1}) . Then, by hypothesis of the Theorem, there exists K such that

$$(\Phi_{\tau_i}^{\theta_K v} \# (\mu^0 \times \mathbb{1}_{A_i}))(\omega) \geq 1 - (\Phi_{\tau_i - T}^{\theta_K v} \# (\mu^1 \times \mathbb{1}_{B_i}))(\omega) - \varepsilon.$$

The function θ_K can be then used to store the mass of μ^0 in ω . The meaning of the previous equation is that the stored mass is sufficient to fill the required mass for μ^1 .

We now define the control achieving approximate controllability at time $T^* + s$ as follows: First of all, using the same strategy as in the Proof of Theorem 1, we can send a part of $\phi_{s-\xi}^{\theta_K v} \# (\mu^0 \times \mathbb{1}_{A_0})$ approximately to $\phi_{-T^*}^{\theta_K v} \# (\mu^1 \times \mathbb{1}_{B_0})$ during the time interval $(s - \xi, s)$. More precisely, we replace T_0^* and T_1^* by $s - \xi$ and ξ in the proof of Theorem 1. Thus, we send the mass of μ^0 contained A_0 near to the mass of μ^1 contained in B_0 . We repeat this process on each time interval (τ_i, τ_{i+1}) for A_i to B_i . Thus, the mass of μ^0 is globally sent close to contained in A_i in time $T^* + s$.

V. EXAMPLE OF MINIMAL TIME PROBLEM

In this section, we give explicit controls realizing the approximate minimal time in one simple example. The interest of such example is to show that the minimal time can be realized by non-Lipschitz controls, that are unfeasible.

We study an example on the real line. We consider a constant initial data $\mu_0 = \mathbb{1}_{[0,1]}$ and a constant uncontrolled vector field $v = 1$. The control set is $\omega = [2, 3]$. Our first target is the measure $\mu_1 = \frac{1}{2}\chi_{[4,6]}$. We now consider the following control strategy:

$$u(t, x) = \begin{cases} 0 & t \in [0, \frac{4}{3}), \\ \psi(2 + (t - \frac{4}{3}), \frac{7}{3} + 2(t - \frac{4}{3})) & t \in [\frac{4}{3}, \frac{5}{3}), \\ \psi(2 + (t - \frac{5}{3}), \frac{7}{3} + 2(t - \frac{5}{3})) & t \in [\frac{5}{3}, 2), \\ \psi(2 + (t - 2), \frac{7}{3} + 2(t - 2)) & t \in [2, \frac{7}{3}), \\ 0 & t \in [\frac{7}{3}, \frac{13}{3}], \end{cases} \quad (23)$$

where $\psi(a, b)$ is defined as follows:

$$\psi(a, b)(x) = \begin{cases} \frac{x-a}{b-a} & x \in [a, b], \\ 0 & x \notin [a, b]. \end{cases} \quad (24)$$

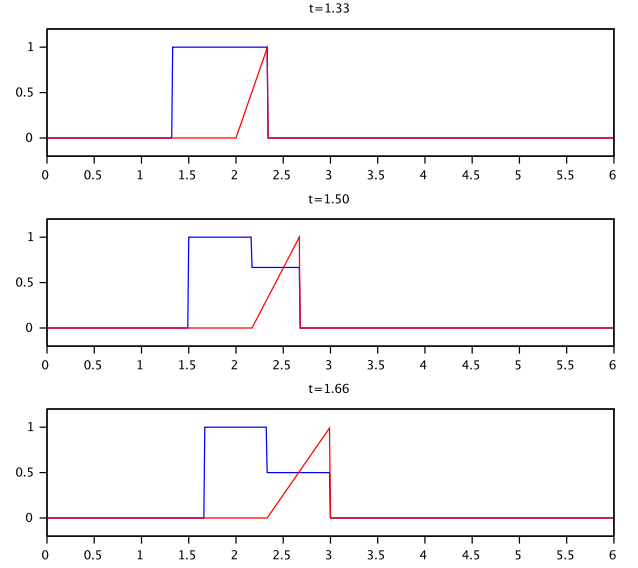


Fig. 3. Blue: density of the measure. Red: control vector field.

The choice of $\psi(a, b)$ given above has the following meaning: the vector field $\psi(a, b)$ is linearly increasing on the interval, thus an initial measure with constant density $k\chi_{[\alpha_0, \beta_0]}$ with $a \leq \alpha_0 \leq \beta_0 \leq b$ will be transformed to a measure with constant density, supported in $[\alpha(t), \beta(t)]$, where $\alpha(t)$ is the unique solution of the ODE

$$\begin{cases} \dot{x} = v + u, \\ x(0) = \alpha_0, \end{cases}$$

and similarly for $\beta(t)$. As a consequence, we can easily describe the solution $\mu(t)$ of (1) with control (23) and initial data μ_0 . For simplicity, we only describe the measure evolution and the vector field on the time interval $[1, \frac{4}{3}]$ in Figure 3. One can observe that the linearly increasing time-varying control allows to rarefy the mass.

Two remarks are crucial:

- The vector field $v + \psi(a, b)$ is not Lipschitz, since it is discontinuous. Thus, one needs to regularize such vector field with a Lipschitz mollificator. As a consequence, the final state does not coincide with μ_1 , but it can be chosen arbitrarily close to it;
- The strategy presented here cuts the measure in three slices of mass $\frac{1}{3}$, and rarefying each of them separately. Its total time is $4 + \frac{1}{3}$. One can apply the same strategy with a larger number n of slices, and rarefying the mass in $[2, 2 + \frac{1}{n}]$ by choosing the control $\psi(2 + t, 2 + \frac{1}{n} + 2t)$. With this method, one can reduce the total time to $4 + \frac{1}{n}$, then being approximately close to the minimal time $T_0 = 4$ given by Theorem 2.

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